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QUANTUM NUMBERS, ATOMIC MASS NUMBERS AND ATOMIC FORCES

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ABSTRACT. By a simple extension of the elementary quantum theory, a relation is derived between the quantum numbers of the orbits of particles and their atomic mass numbers. The force between the constituents of a system of atomic dimensions is derived in terms of the mass numbers of the constituents and the distance between them. There follows a similar derivation of the energy of the system. The application to the cases of the hydrogen-like atoms and the deuteron is considered,

INTRODUCTION

The quantum numbers are characterized by their integral values, and so are the atomic mass numbers. By atomic mass number we mean the atomic weight expressed in terms of the mass of the proton or the neutron as the unit. The atomic mass numbers are known to be integers from the results of the experiments with the mass-spectrograph; and, therefore, they are taken to represent the number of protons and neutrons in the atomic nucleus.

The fundamental quantum relation in the elementary Bohr theory of the hydrogen atom is

$$2\pi mvr = nh,$$

where m is the mass of the electron, r the radius of the orbit, and v the velocity of the electron in the orbit. n is the quantum number, and it is an integer. We may expect a similar relation to hold for the electron's partner, the proton, which is also moving in a circular orbit round the common centre of gravity. Generalising still further, consider any two such particles, A and B, of atomic mass numbers m_1 and m_2, bound together by a common force of attraction, and revolving in circular orbits round the common centre of gravity. Let r_1 and r_2 be their distances from the centre of gravity, so that the distance between them is $r_1 + r_2 = r$; and let ω be the common angular velocity of the two particles. Their masses would be Mm_1 and Mm_2, where M is the mass of the proton. Then, we have the following relations.

$$m_1 r_1 = m_2 r_2 \tag{1}$$

$$2\pi M m_1 r_1 v_1 = 2\pi M m_1 \omega r_1^2 = n_1 h, \tag{2}$$

and

$$2\pi M m_2 r_2 v_2 = 2\pi M m_2 \omega r_2^2 = n_2 h, \tag{3}$$

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where n_1 and n_2 are the quantum numbers. Substituting in (1) from (2) and (3), we have

$$\frac{n_1 h_1}{2\pi M\omega r_1} = \frac{n_2 h}{2\pi M\omega r_2} \quad \dots \quad (4)$$

Or,

$$\frac{n_1}{r_1} = \frac{n_2}{r_2} \quad \dots \quad (5)$$

which gives

$$\frac{r_1}{r_2} = \frac{n_1}{n_2} = \frac{m_2}{m_1} \quad (6)$$

This can be also expressed as

$$n_1 = sm_2,$$

and

$$n_2 = sm_1. \quad (7)$$

If n_1, n_2, m_1, and m_2 are to be integers, we can see that s can have only a discrete set of values. These would be, (*i*) any integer ; (*ii*) a fraction equal to the reciprocal of any factor common to m_1 and m_2 ; which we may call $1/f$ in general ; (*iii*) all numbers expressed by $n + 1/f$, where n is an integer. Thus, for example, for $m_1 = 20$ and $m_2 = 40$, the values of s would be, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{20}$, 1, $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{5}$, $1\frac{1}{10}$, $1\frac{1}{20}$, 2, $2\frac{1}{2}$, $2\frac{1}{4}$, etc. We may also put it this way : n_1 and n_2 are integers because m_1 and m_2 are integers. n_1 and n_2 express the multiples of the number of protons and neutrons in the particles, or their numbers in the equal sub-groups which they can form in the two particles.

The moment of inertia of the system would be given by

$$J = Mm_1 r_1^2 + Mm_2 r_2^2 = \frac{h}{2\pi\omega}(n_1 + n_2) \text{ from (2)}$$

$$= \frac{hs}{2\pi\omega}(m_1 + m_2) \text{ from (7)} \quad \dots \quad (8)$$

The energy of the system would be given by

$$\tfrac{1}{2}J\omega^2 = \frac{h\omega s}{4\pi}(m_1 + m_2) \quad \dots \quad (9)$$

The centrifugal force acting on A is

$$\frac{Mm_1 v_1^2}{r_1} = Mm_1 r_1 \omega^2 \quad \dots \quad (10)$$

and that on B is, similarly,

$$Mm_2 r_2 \omega^2$$

Each of them is equal to the force of attraction, F, between A and B, and the two balance, holding A and B together. Therefore,

$$F = \frac{M\omega^2}{2}(m_1 r_1 + m_2 r_2)$$

$$=\frac{M\omega^2}{2}\left(\frac{m_1 r_1^2}{r_1}+\frac{m_2 r_2^2}{r_2}\right)$$

$$=\frac{h\omega}{4\pi}\left(\frac{n_1}{r_1}+\frac{n_2}{r_2}\right), \text{ from (2) and (3)}$$

$$=\frac{h\omega}{2\pi}\,\frac{n_1+n_2}{r_1+r_2}, \text{ from (5)}$$

$$=\frac{h^2}{4\pi^2 J}\,\frac{(n_1+n_2)^2}{(r_1+r_2)}, \text{ from (8)}$$

$$=\frac{h^2}{4\pi^2 M}\,\frac{(n_1+n_2)^2}{(r_1+r_2)}\,\frac{1}{(m_1 r_1^2+m_2 r_2^2)}$$

$$=\frac{h^2}{4\pi^2 M m_1 r_1}\left(\frac{n_1+n_2}{r_1+r_2}\right)^2, \text{ from (1)}$$

$$=\frac{h^2 s^2}{4\pi^2 M m_1 r_1}\left(\frac{m_1+m_2}{r_1+r_2}\right)^2, \text{ from (7)}$$

$$=\frac{h^2 s^2}{4\pi^2 M m_1 m_2}\left(\frac{m_1+m_2}{r_1+r_2}\right)^3, \text{ from (6)}$$

$$=\frac{h^2 s^2}{4\pi^2 M m_1 m_2}\left(\frac{m_1+m_2}{r}\right)^3 \qquad \ldots \quad (11)$$

This is an expression of the force between A and B in terms of their masses and the distance between them, but it cannot be called a gravitational force for that reason, as it only gives the centrifugal force upon A or B. It is only a general expression of the force between A and B, irrespective of its nature. If this is known, r can be expressed as a function of s; *i.e.*, it will have discrete values; and it is only for these sets of values of s and the corresponding values of r that (11) holds.

The energy of the system would be given by

$$E=\tfrac{1}{2}J\omega^2=\tfrac{1}{2}\frac{(J\omega)^2}{J}=\frac{h^2(n_1+n_2)^2}{8\pi^2 J}$$

$$=\tfrac{1}{2}F.r=\frac{h^2 s^2}{8\pi^2 M m_1 m_2}\,\frac{(m_1+m_2)^3}{r^2} \qquad \ldots \quad (12)$$

If A and B are spinning with angular velocities ω_1 and ω_2 and J_1 and J_2 are moments of inertia about the axes of spin, the total moment of momentum would be

$$J\omega = J_1\omega_1 + J_2\omega_2$$

By a simple extension of (8), this becomes,

$$\frac{h}{2\pi}\{s(m_1+m_2)+s_1 m_1+s_2 m_2\} \qquad \ldots \quad (13)$$

where s_1 and s_2 are integers.

Also,

$$E = \tfrac{1}{2}J\omega^2 + \tfrac{1}{2}J_1\omega_1^2 + \tfrac{1}{2}J_2\omega_2^2$$

and

$$\tfrac{1}{2}J_1\omega_1^2 = \frac{h^2 m_1^2 s_1^2}{8\pi^2 M m_1 k_1^2} \qquad (14)$$

where k is the radius of gyration about the axis of spin.

APPLICATIONS

(*a*) *The Hydrogen-like Atom*

This consists of a nucleus of atomic mass number m_1, and a positive charge Ze, with an electron of charge $-e$ and mass number $m_2 = 1/1839$ revolving round it. The centre of gravity practically coincides with the nucleus. This is, of course, a departure from the picture of the general system described above in that one of the atomic mass numbers is not an integer. The limitations set out above for the values of s will no longer hold in this case, the only limitation being the integral values of n_1 and n_2; *i.e.*, of sm_1 and sm_2. From (11), the force between the nucleus and the electron is

$$F = \frac{h^2 m_1^2 s^2}{4\pi^2 M m_2 r^3} \qquad (15)$$

The force is the Coulomb force of attraction, $\dfrac{Ze^2}{r^2}$.

Therefore,

$$\frac{h^2 (sm_1)^2}{4\pi^2 M m_2 r^3} = \frac{Ze^2}{r^2} \qquad (16)$$

giving

$$r = \frac{h^2 n_1^2}{4\pi^2 m Z e^2} \qquad (17)$$

where $Mm_2 = m$, the electronic mass.

The energy of the system, as also seen from (12) and (17) is

$$E = \frac{Ze^2}{2r} = \frac{2\pi^2 m Z^2 e^4}{h^2 n_1^2} \qquad (18)$$

which is the familiar Bohr formula.

(*b*) *The Deuteron*

Here $m_1 = m_2 = 1$, and hence the values of s are all integral. From (12), the energy of the system is

$$E = \frac{h^2 s^2}{\pi^2 r^2 M} \qquad \cdots \quad (19)$$

The potential function is very probably of the form $\frac{Ve^{-\lambda r}}{r}$. The general procedure would be to solve the equation

$$\frac{h^2 s^2}{\pi^2 r^2 M} = \frac{Ve^{-\lambda r}}{r}$$

or

$$\log r - \lambda r = \frac{h^2 s^2}{\pi^2 VM} \qquad \cdots \quad (20)$$

to obtain r as a function of s; and using these values of r and s to obtain the energies for the different levels. As this does not seem feasible, we may take particular values of s, find the corresponding value of r by trial or otherwise, and calculate the energy. Thus, for $s = 1$,

$$re^{-\lambda r} = \frac{h^2}{\pi^2 VM} \qquad \cdots \quad (21)$$

Assuming the values $\frac{1}{\lambda} = 3.8 \times 10^{-13}$, and $V = 7.7 \times 10^{-18}$ as deduced elsewhere, (Soonawala, 1942) this becomes

$$\log r - 1.1 \times 10^{12} \times r = -12.5 \qquad \cdots \quad (22)$$

where $\log r$ is to the base 10. We can see that the maximum value of r is attained for $r = 1/1.1 \times 10^{12}$, and is equal to about -13. This nearly but not quite fits into (22). Assuming it to be a close enough approximation, $E = 2.6 \times 10^{-6}$ erg, which is near enough to the accepted value. This is not altogether surprising as the values of λ and V were originally deduced from a similar value of E.

We can hardly be justified in improving the fit by selecting a suitable value of λ, as this would involve a corresponding correction for V; but our present knowledge of the nucleus is too meagre. However, it would illustrate some interesting features. Thus assume $\frac{1}{\lambda} = 4.3 \times 10^{-12}$ leaving V the same as before. Then,

$$\log r - 1 \times 10^{11} \times r = -12.5 \qquad \cdots \quad (23)$$

The maximum of $\log r - 1 \times 10^{11} . r$ is reached for $r = 10^{-11}$ as well as $r = 10^{-12}$, and is about equal to -12. But, while with $r = 10^{-12}$ the second term is of small significance, it is not quite so insignificant when $r = 10^{-11}$.
For $s = 2$, we have

$$\log r - 1 \times 10^{11} r = -11.9 \qquad \cdots \quad (24)$$

As before, with $r = 10^{-12}$ approximately, the second term is of small importance, and then r varies as the second, or almost the second, power of s. The energy varies as $\frac{s^2}{r^2}$, as seen from (12), or nearly as $\frac{1}{s^2}$. On the other hand, if values of r are chosen which make the term λr of greater significance than log r, it can follow that the energy of binding would increase with s.

Another important feature which becomes illustrated is a certain upper limit set on s. This happens because the right hand side of (20) can, for large enough values of s, attain a value corresponding to which no value of r is possible, the maximum of $\log r - \lambda r$ exceeding $\frac{h^2 s^2}{\pi^2 VM}$. The energy levels would then be few and discrete. The tightest binding is for $s = 1$, and diminishes with higher values of s.

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REFERENCE

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